GRADE 10 STUDENTS' FACILITY WITH RATIONAL ALGEBRAIC FRACTIONS IN HIGH STAKES EXAMINATION: OBSERVATIONS AND INTERPRETATIONS

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In this study students' facility with rational algebraic fractions is explored in a high stakes examination context. The presences of sub-constructs within fractions function as an additional complexity of rational fractions. We confirmed that the categories of student error that we found, mirrored a landmark previous study done by Figueras et al.(2008), but only in part; we found in addition that a number of students committed an "algebraic equation" error in that they converted rational fractions to algebraic equations and then tried to solve them. This study shows that the presence of visual cues in problems involving algebraic fractions act as distractive stimuli and we direct our analysis to these sub-constructs to deconstruct student difficulties with algebraic fractions. Our study makes useful recommendations for teachers and teaching.

INTRODUCTION

The teaching and learning of fractions is one of the most problematic areas in the Senior Phase (SP) and Further Education and Training Phase (FETP) school grades. In South Africa the SP refers to Grades 7, 8, and 9, whereas the FETP refers Grades 10, 11, and 12 (Department of Basic Education, 2011). Educational research regards fractions as a challenging concept in the curriculum, and alleges that "… problems in understanding fractions persist into adulthood, with moderate to severe consequences for everyday and occupational decision-making" (Ross & Bruce, 2009, p.713). During the past three decades, research in mathematics education has identified that the multifaceted nature of fractions is a major contributing factor to the core difficulties experienced by teachers and students during the teaching and learning of fractions. This multifaceted construct is made up of five interrelated sub-constructs as follows: *part-whole* (which gives rise to the notion of partitioning an object or set into smaller equal sections), *ratio* (gives the natural way of show casing the procedures associated with finding the equivalent fractions), *operator* (this is key to developing an understanding of the multiplication operations of fractions), *measure* (refers to the idea that fractions are identified by their size, viz., the distance from a point of reference – this gives rise to the proficiency of ordering fractions on a number line), and *quotient* (underscores the notions that any fraction can be represented as a division operation, it is about fair-sharing, and the fact that the denominator is bigger than the numerator or vice versa does not matter). Learning fractions is difficult because it requires a deep understanding of all the latter sub-constructs. (Charalambous & Pitta-Pantazi, 2007; Behr et al., 1993; Kieren, 1993; Lamon, 1993, 1999, 2001; Marshall, 1993; Ross & Bruce, 2009).

Failure of students to conceptually understand these sub-constructs in the SP school grades leaves students with reduced chances for learning these skills in FETP school grades due to the congested nature of the curriculum. One key factor which contributes to the students' inability to learn fractions is the claim that teachers are not well equipped, through teacher education training programmes, to effectively teach the concept of fractions. As a result, rote learning appears to be the norm during teaching and learning of fractions (Gowan, et al., 1990). From a historical perspective, research has also shown that students in the SP and FETP school grades harbour a dislike of fractions primarily because they find "that fractions were irrelevant to the solution of mathematics problems in anyone's daily life" (Groff, 1996, p.177). In support for Groff (1996), Van Hiele (1986, p. 211) states "to be honest, we should admit that being able to calculate with fractions has no practical utility". These arguments put forward by Groff (1996) and Van Hiele (1986) support historical justifications for students to not engage with the learning of fractions.

Perceptions may, however, have changed during the past three decades (Usiskin, 2007). Usiskin (2007, p.370) underscores the importance and justification for inclusion of fractions in the formative years of mathematics education:

> The realization that fractions represent division and constitute the most common way in which division is represented in algebra has caused a demand for increasing competence in fractions by all those for whom algebra skills are important.

Whilst there is a common agreement that fractions are difficult for students to conceptualise, there is generally strong support for the critical importance of teaching fractions to SP students. The justifications for doing so are many. Firstly, formal and prolonged exposure to the teaching and learning of fractions equip students with the pre-requisite necessary skills for "partitioning" that will enable them to understand and describe real world phenomena (Groff, 2006). Secondly, fractions are the gate for higher mathematics – this means that if students have a weak background in fractions they are likely to experience challenges in understanding concepts in algebra, higher-order mathematics such as number theory, and calculus (Aliberti, 1981; Karim et al., 2010). Brown & Quinn (2007b, p.9) claim that "to solve rational equations and simplify rational fractions it is necessary to apply generalised common fractions concepts". The point here is that students whose understanding of fractional constructs are weak are likely to find rational algebraic fractions concepts difficult (Brown & Quinn, 2006, 2007a, 2007b; Usiskin, 2007; Groff, 2006). In contrast, not all mathematicians and educators are convinced about the intrinsic value of fractions when solving real life problems or as the pre-requisite for higher order mathematics. The quotient sub-construct, argue Enright (1998) and Van Hiele (1986), is difficult to implement because dividing objects in perfectly equal parts is practically impossible.

Hence, teachers will find it difficult "to make up genuine problems that can be solved through the manipulation of fractions" (Groff, 2006, p.552). Figueras et al. (2008) suggest that rational fractions pose another particular problem: it is difficult to give concrete examples to go with the rational algebraic fractions introduced, consequently students may also struggle to generate examples themselves.

Whilst we acknowledge the concerns raised in the rebuttals that proficiency in fractions does not translate to successful study of algebra, in this article we argue that students with insufficient knowledge of fractional sub-constructs from the SP school grades are more likely to experience learning difficulties when simplifying rational algebraic fractions they encounter in the FETP curriculum.

Rational algebraic fractions contain all the difficulties of ordinary fractions, but in addition it includes those difficulties usually associated with algebra generally. These include addition and subtraction involving like and unlike terms, multiplication and division. These operations take place *inside* each algebraic fraction and thus mirror some of the sub-constructs of ordinary fractions. Further, other difficulties include the addition of algebraic fractions, the use of a lowest common denominator and the operations involved in adding these terms, in much the same way that ordinary fractions are added.

CLASSIFICATION OF ERRORS ON SIMPLIFYING RATIONAL EXPRESSIONS

In addition, and acting in a complementary fashion to the idea of sub-constructs, Figueras et al. (2008), found seven categories of student error when testing for students' understanding of simplifying rational expressions. They administered the tests after the students had specifically been taught rational expressions and arrived at the following categories: (a) *Cancellation* – using a constant term, variable, coefficient that was present in both numerator and denominator and was cancelled. For example, if we consider the category "cancellation" we may observe that students usually do not understand that in any fraction if the denominator and numerator are at the same time multiplied or divided by the same number, the original fraction does not change.

For example: $\frac{2(5)}{3(5)}$, $\frac{8 \div 4}{12 \div 4}$, and $\frac{2(3x+4)}{2(x+y)}$ will be the same as $\frac{2}{3}$, $\frac{8}{12}$ and $\frac{3x+4}{x+y}$ respectively. Secondly, however, if the same number is added or subtracted to both the numerator and denominator of any fraction, the resulting fraction will be different for the original one. For example: $\frac{5+3}{7+3}$, $\frac{5-2}{8-2}$, and $\frac{4a+3}{4b+1}$ will not be the same as $\frac{5}{7}$, $\frac{5}{8}$ and $\frac{a}{b}$ respectively.

(b) *Partial division*: division taking place but only between some of the terms. (c) *Like term error 1* - students committing an error other than division (usually subtraction) between like terms in the numerator and denominator. (d) *Like term error 2* - performing a mistaken operation in the numerator or denominator (for example adding wrong). *Linearization* - breaking up a rational expression with a compound into two separate rational expressions. (e) *Defractionalisation* - transformation of fraction with unity numerator to a non-fraction, for instance, $\frac{1}{3}$ becomes 3. (f) *Equationisation* - transforming a rational fraction into a rational equation.

The categories "like term error (1) and (2)" above can clearly be further delineated into still more categories if one choses. For example, one can distinguish between the "mistaken operations" which students make in the numerator and denominator, or for that matter in the *kinds* of addition which students were making which were wrong. There is also a conflation of addition and multiplication when working with rational fractions, much like the conflation which happens when working with exponents. These categories sound more like the sub-construct of fractions discussed above.

Thus underlying the concept of sub-constructs and also the prevalence of particular forms of errors are operations and understanding of basic algebra, involving addition and subtraction of like and unlike terms, multiplication and division of algebraic terms and so on. Therefore, as other researchers concur, the area of simplifying rational expressions in algebra research is a resource with a rich supply of data as we also experienced. It is claimed by some that these kinds of errors are not focussed on by teachers because they do not realise that students struggle to cope with the fundamental concept of cancellation, for example, among other difficulties. In fact it is claimed that student apply rote learning when simplifying fractions and rational algebraic fractions as a matter of course (Grossman, 1924).

PURPOSE OF THE STUDY

One of the key objectives of learning algebra at the key FETP in Grade 10 is to manipulate algebraic expressions by: "Simplifying, adding and subtracting algebraic fractions with denominators of cubes (limited to sum and difference of cubes)" (Department of Basic Education, 2011, p.13). Drawing on the theoretical model of the five sub-constructs on fractions and rational numbers presented by Charalambous and Pitta-Pantazi (2007), and the categories of errors as outlined by Figuera, et al. (2008), this study investigated the proficiencies of grade 10 mathematics learners in simplifying rational algebraic fractions in a high-stakes end of year examination. In particular, the study sought answers to the following research question: How proficient are grade 10 mathematics learners in the simplification of rational algebraic expressions? The findings of this study will serve as a platform for further research that will provide insights into high school mathematics students' understanding of simplifying rational algebraic fractions.

In addition, the study can inform classroom practice of mathematics teachers, teacher educators of mathematics, and teachers of post- secondary school mathematics on the teaching and learning of rational algebraic fractions. It is critical for students to have an understanding of rational expressions, especially students who intend to successfully study majors in science, technology, engineering, and mathematics (STEM).

METHODS

The data collected consists of end of year test scripts of the grade ten mathematics classes in schools involved in the Local Evidence Driven Improvement Mathematics Teaching and Learning Initiative (LEDIMTALI) Project. This project, a First Rand Mathematics Chair project based at University of Western Cape, involves: in-servicing mathematics teachers from several schools in a teacher development programme, improving understanding and teaching of mathematics in schools. In particular, student responses to question 2.3 of the examination paper is analysed in this study.

The problem (question 2.3) given to the students contained three rational fractions, namely,

(a) $\frac{x}{x+y}$, (b) $\frac{x^2+y^2}{y^2-x^2}$, (c) $\frac{2x+y}{x+y}$.

The question was to prove that: $\frac{x}{x+y}-\frac{x^2+y^2}{y^2-x^2}=\frac{2x+y}{x+y}$. There was a typographical error in fraction (b) as it was supposed to read: $\frac{x^2-y^2}{y^2-x^2}$, however, this typographical error does not impact on the overall results of this study whose main focus was not to assess whether students can prove the equivalence of the rational algebraic fractions, but to show students' proficiencies in simplifying rational algebraic fractions.

In all three algebraic fractions, the *visual cues* provided tempting examples of like terms in the denominator and numerator which could be cancelled. In fraction (a), the denominator contains a sum of two unlike terms, and the visual cue of an x in the numerator and denominator. A student could thus decide to add the terms in the denominator, or cancel the x's. In fraction (b) there are two sub-constructs, an addition of two squares in the numerator and the difference between two squares in the denominator. The visual cues of the x's and y's may cause the student to use the cancellation error twice. Or the student may recognize the case of the two squares and factorise (but factorising the sum of squares would be seen as an error of factorization). In the third case (c), there are again two sub constructs, both sums of unlike terms. Again the visual cues of the presence of x's and y's may lead to the cancellation error.

The overall picture (did the student see the wood as well as the trees?) is the requirement to prove that (a) minus (b) equals (c). The student must perform a subtraction (a minus b), *simplify* the quotient and *show* that this is equal to the third algebraic fraction, (c).

Thus the problem contains a myriad of operations, any number of which can go wrong. With this kind of procedural problem is that it is very difficult to determine why things went wrong if they do go wrong. In such cases we must make speculative judgement calls. Diagnoses require detailed exploration of all possible options.

DATA ANALYSIS

From an initial sample of 39 students (that is, one batch containing students from one school) we created the following categories: 14 simply wrote down the question, and there was no attempt to solve it, 15 left out the question (did not write it down). We were left to analyse 10 scripts. The sample of ten scripts was further broken down as follows: four were analysed to assess the extent to which lack of facility with sub-constructs led to errors in solving the problem posed. All ten were analysed in terms of the categories of errors. As it turned out, the use of the cancellation error was the most frequent in the four sample scripts analysed for sub-construct errors.

Sub-constructs

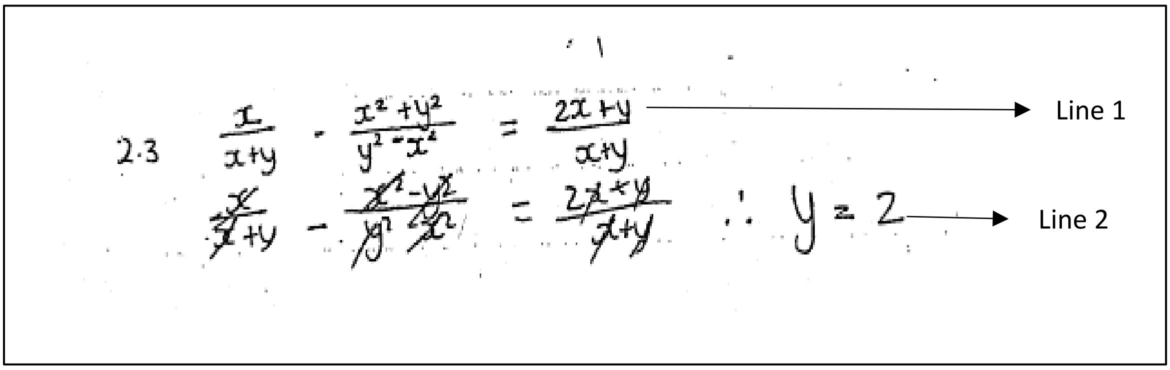

Figure 1: Sub-construct and cancellation error – example 1

In figure 1, the student applied the cancellation error thrice and was left with: a y in the denominator, a 2 in the numerator in the third algebraic fraction and numerous negative signs (notice there was no quotient "1" after "like" terms have been "cancelled"). The student then ends up with the solution: y=2. The *visual cues* appeared to have dominated this student's approach to solving the problem. There was no attempt to deal with the sub-constructs.

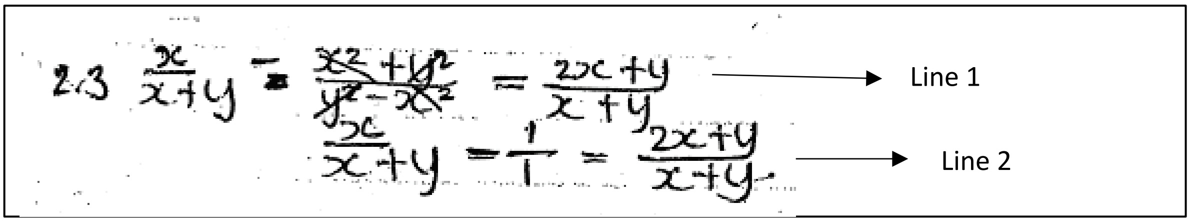

Figure 2: Sub-construct and cancellation error – example 2

In figure 2, the student applied the cancellation error to the second algebraic fraction only, was left with an answer for that operation, namely 1, and left the problem in that shape. For reasons that can only be brought out during an interview, the student did not cancel any of the other terms. Again no attempt was made to engage the sub-constructs.

In figure 3, the student correctly joins the two algebraic fractions (thereby demonstrating understanding for addition of algebraic fractions), and, at the same time ends up with a new algebraic fraction on the right hand side, notably with a denominator equal to that on the left hand side. An alternative route for the student at that point was to equate the two numerators, especially after having made the denominators equal, presumably for that purpose.

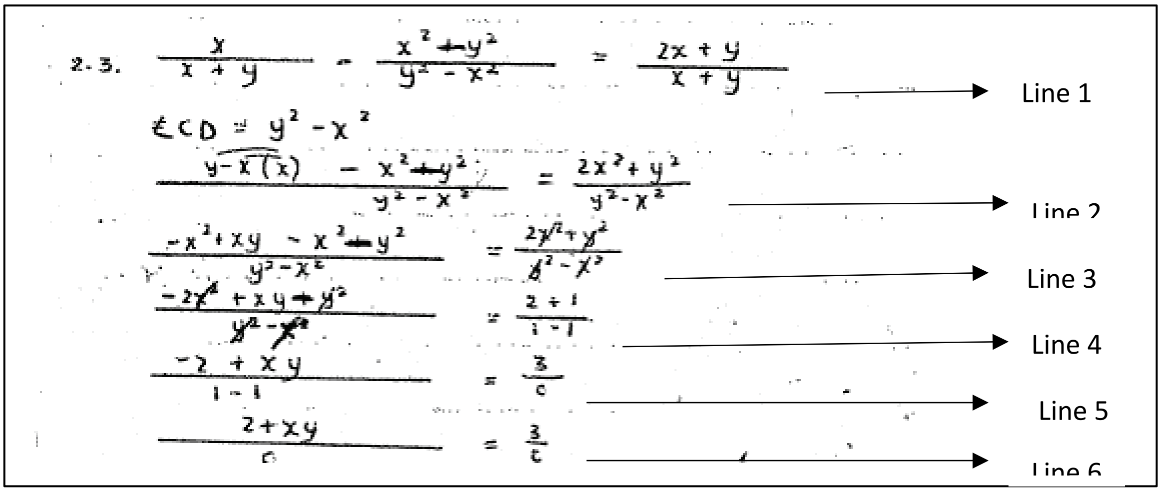

Figure 3: Sub-construct and cancellation error – example 3

Again further probing of the student may provide more information. Extensive use is made of the sub constructs to factorise and add algebraic fractions using a lowest common multiple approach.

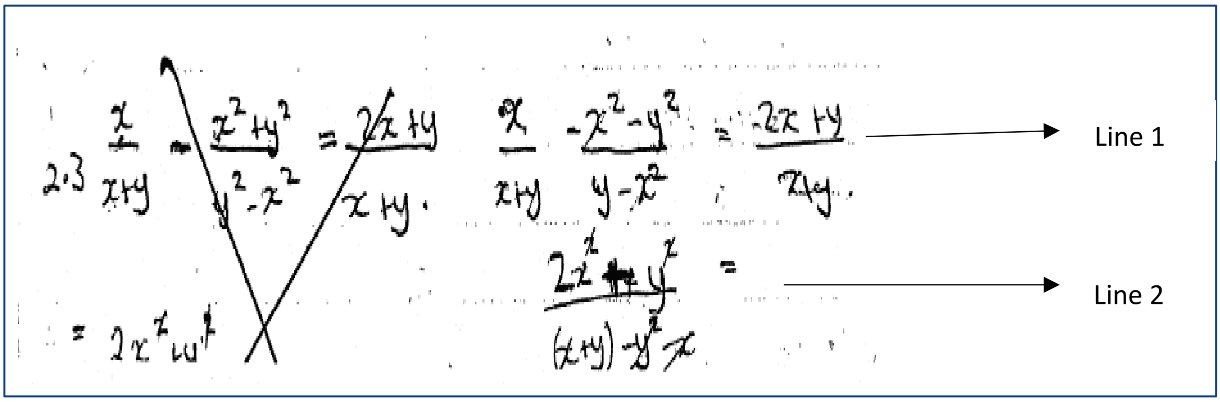

Figure 4: Sub-construct and cancellation error – example 4

In Figure 4, there is an example of the type of error which has been called type 2 (T2) in Figueras, et al. (2008). From the point of view of sub-constructs it is not clear from the script what methods the student employed to bring the two algebraic fractions together. This is then followed by cancellation.

CATEGORIES OF TYPES OF ERRORS

In this section we discuss four categories of the errors found in the ten scripts that we analysed, and how they manifested in the responses. These are: “cancellation” errors, “performing a mistaken operation” error, “grouping like terms” error, and “factorisation” errors. The “performing a mistaken operation” (T2) error fits better in the “errors committed in the sub-constructs” as detailed above, but are repeated here, as part of the original classification of Figueras, et al. (2008).

From the ten students whose scripts were analysed we observed that: 4 committed the T2 error Figueras et al. (2008) performing a mistaken operation in the numerator and/or denominator, 5 had cancellation errors (this included other errors not classified by the Figueras et al. (2008), 2 had a *factorisation* error (they responded the same way) but cancelled properly afterwards, and 4 committed what we termed an algebraic equation error – they converted the rational expressions into an algebraic equation (varied forms).

Cancellation errors

Typically, the cancellation error involved students cancelling like terms in the numerator and denominator, without consideration for the rational fraction as a whole, much as was documented by Figueras et al. (2008). This was covered in the sub-constructs section.

Performing a mistaken operation error

The “like term error 2” which involved mistakes in which students simply joined variables as in $2x + y = 2xy$ treating the expression $2x + y$ within the wider problem as if it was a multiplication problem, occurred a number of times. For examples of this type of error see figure 5, lines 2.

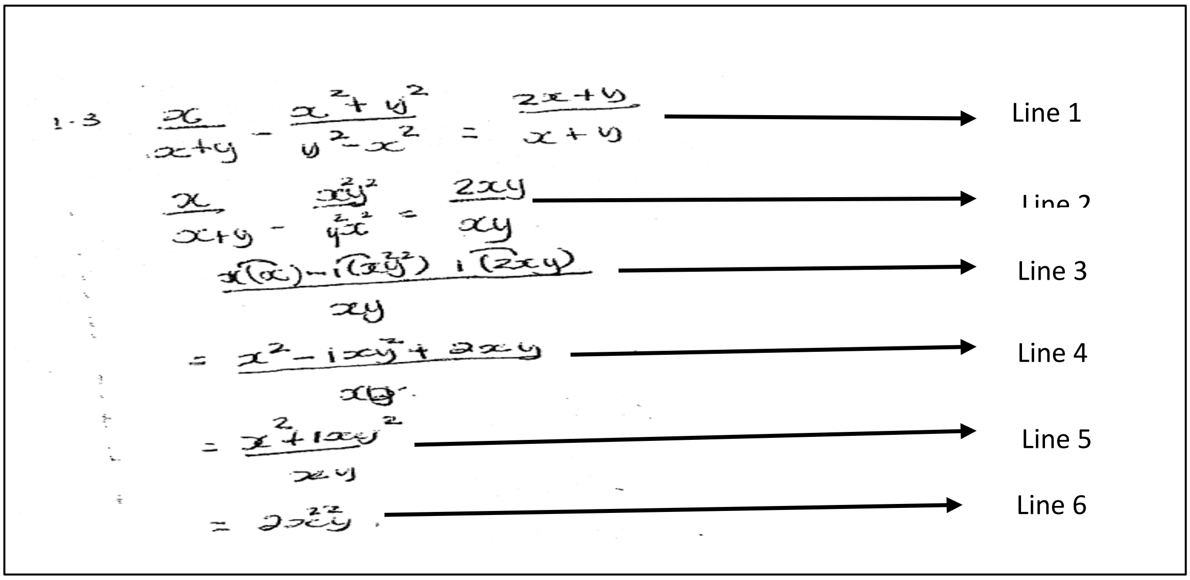

Figure 5: Mistaken operation error

The sub-text of student responses here appears to be that the student knows to bring all the rational fractions to a state where they could be joined under one denominator and then calculated – the students certainly aimed for that. Further errors occur once there is the comfort of joining the fractions under one denominator – for example calculation errors classified in type 1. In the main, once the denominator is created (and is false) the students' work is so wrong that it cannot be saved by another contrived step.

Grouping like terms error

Of particular interest, and not mentioned in the Figueras et al. (2008) list is the students' use of an algebraic equation in place of the rational fraction, which is then solved.

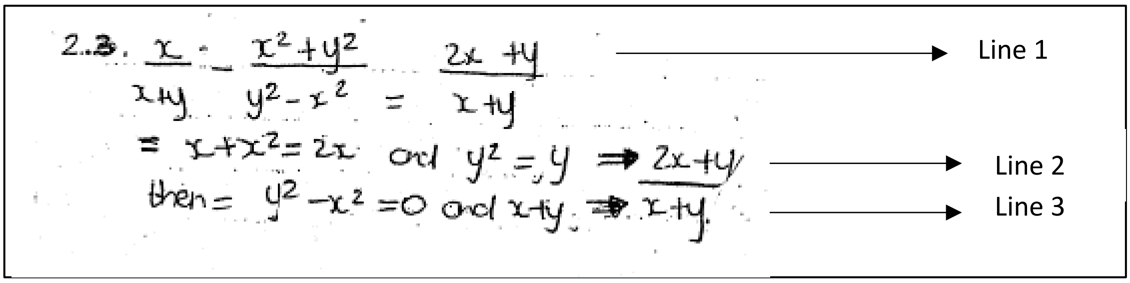

Figure 6: Grouping like terms error

There is either logic, which on the face of it has no sense (but is logical anyway) (see line 2, when each part of one numerator on the left hand side is "added" and then equated to the numerator in the other side), while some mixed logic is applied to the denominator (line 3: $y^2 - x^2$ is equated to x *plus y*, both groups in denominators of their respective fractions.

Factorisation errors

The existence of factorisation errors (figure 7) meant that at least some students thought that they had to factorise to solve the problem. The difficulty of the second rational fraction is that $y^2 - x^2$ is an example of the difference between two squares and can be factorised using a formula, $y^2 + x^2$ could not be factorised. However the *visual cue* in this case may have been that the two expressions were placed in a rational fraction – thus *the temptation to cancel* was presented, but only once the students had factorised. *Is this a case where knowledge of the outcome leads to certain forms of behaviour?* In both cases the students factorised that rational fraction in order to cancel like terms. The cancellation was proper but the factorisation was not. This indicated that these two students knew what kind of cancellation was not allowed and what kind was *desirable*.

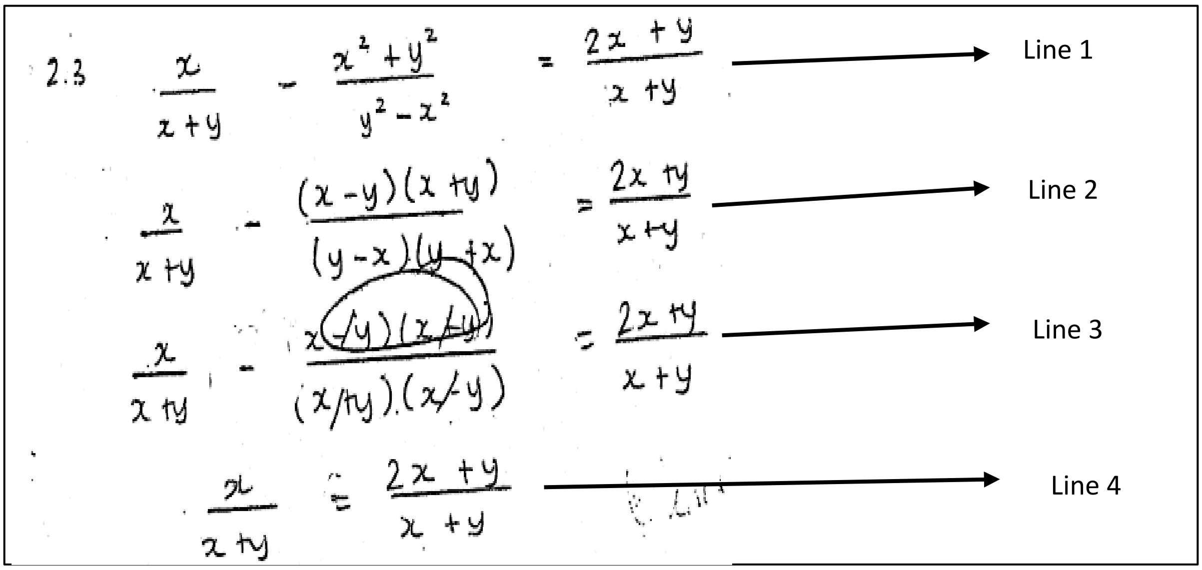

Figure 7: Factorisation error

DISCUSSION OF THE FINDINGS

In reviewing the overall mathematical behaviour of the students who attempted the question under discussion, a number of issues lend themselves to speculation. For example, what is the role of the right hand side in questions in which students have to prove an equation or identity? In this case, the students had to prove that a left hand side, which was hugely complicated, equalled a right hand side, which looked more manageable. Some were clearly totally out of their depth and simply left the question out or wrote it down but did not attempt it. Of particular note is that one student scored in the top percentile for the test and yet left out question 2.3. There is no evidence that the student attempted the problem and discovered that it could not be solved.

Our assumption is that the student found the problem to be too difficult to attempt in the examination context. Others attempted to deal with the complications which the left hand side of 2.3 offered, using a number of documented error-strewn strategies, to little or no effect, but always striving to bring or force the left hand side to be the same as the right. In a few cases, the students attempted to "solve" the equation, providing a value as solution for x or y.

There is some research which shows that certain students are not ready to accept arithmetic expressions as concepts in their own right, for example 2+3; that they expect that such a concept ("sum") is actually a process (addition) which has as its end goal a mathematical object, namely 5 (which is at the same time a mathematical *symbol!).* Gray and Tall (1994) introduced the notion of a *precept*, that is, a mathematical object that was at once a process and a concept. They postulated that how the students viewed the mathematical object would influence what they would try to do with it. Translated to algebraic expressions, the algebraic fraction $\frac{x}{x+y}$ is at once an algebraic fraction $\frac{A}{B}$ as well as, in the denominator, an algebraic expression. We thus have a procept within a procept: the algebraic fraction consisting of two separate expressions A and B, and, in the case of B, another precept (addition of x and y and the sum "$x+y$"). The result is that some students may lean towards a procedural way of thinking about mathematics while others will develop a perceptual way of thinking about mathematical problems. Solving complex mathematical structures such as algebraic fractions can be a dividing line for these different levels of thinking about and doing mathematics.

RECOMMENDATIONS

It is clear that algebraic fractions are multi-complex and contain a myriad of difficulties for students in the early stages of learning algebra. The existence of several sub constructs within any one construct of an algebraic fraction and the difficulties of seeing algebraic fractions as entities in their own right (procepts) and not only as procedures all impact on students' abilities to navigate procedural mathematical problems.

For diagnostic purposes, it is perhaps better suited to use this kind of problem with a set (group) for whom basic operations in algebra has been achieved at the required level of competency. The concatenation of levels of algebraic development cannot be very useful in a developmental context.

We questioned whether the cases where knowledge of the end result lead to certain forms of behaviour, especially where students have to prove that one side of an equation equals another. This could be further explored.

CONCLUSION

In conclusion, the findings of this study have implications for the teaching and learning of rational algebraic fractions in secondary schools. This study has demonstrated the existence and the extent of students' conceptions, and misconceptions related to rational algebraic fractions. The nature of the students' difficulties with respect to rational algebraic fractions is conceptual – students may be operating on rational algebraic fractions without necessarily understanding or justifying what they are doing. Given that the current study focussed on the analysis of the examination scripts, we posit here that additional research, that includes the qualitative analysis of students' written examination scripts and tasks-based focus groups interviews, is required to gain further insights into the students' deficiencies with rational algebraic fractions. Finally, we are of the opinion that if understanding rational algebraic fractions is fundamental to successful studying of advanced mathematical concepts required in STEM, then deficiencies in rational algebra fractions need to be identified and addressed early.

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